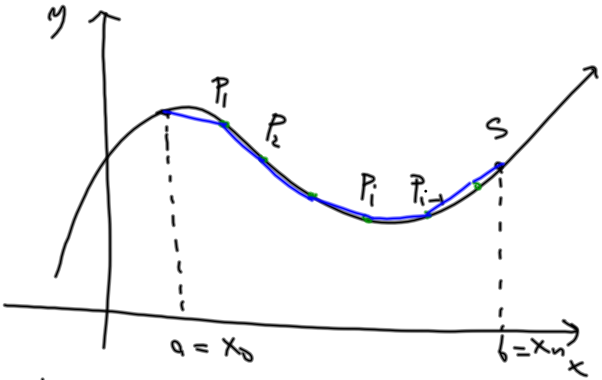
7.4 Length of an arc and area of surfaces of revolution

1 Length of an arc

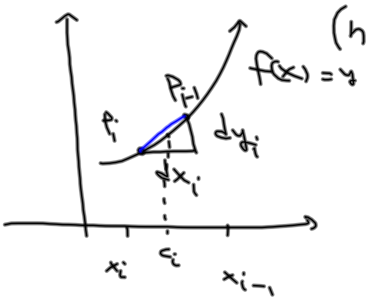


$f(x)$ f is a smooth and continuous function over $[a,b]$

The length of f over $[a,b]$: S

let $|L_i| = \overline{P_i P_{i-1}}$

$$\lim_{\substack{\|P\| \to 0 \\ (n \to \infty)}} \sum_{i=0}^{n} |L_i| = S$$



$$\longrightarrow ds_i^2 = dy_i^2 + dx_i^2$$

$$\longrightarrow \left(\frac{ds_i}{dx_i}\right)^2 = 1 + \left(\frac{dy_i}{dx_i}\right)^2$$

$$\longrightarrow \frac{ds_i}{dx_i} = \sqrt{1 + \left(\frac{dy_i}{dx_i}\right)^2}$$

$\frac{dy}{dx} = f'(x)$

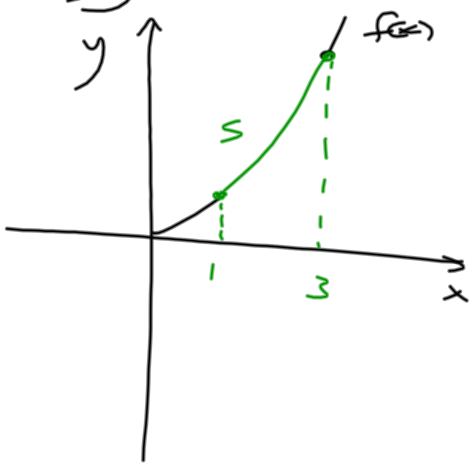
$$\longrightarrow \frac{ds_i}{dx_i} = \sqrt{1 + [f'(c_i)]^2} \quad \text{where.}$$

$c_i \in [x_{i-1}, x_i]$

$$\boxed{\longrightarrow S = \int_a^b \sqrt{1 + [f']^2}\, dx}$$

length of f over $[a,b]$

Eg: Find the length of $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ over $[1, 3]$



Solution:

$$S = \int_1^3 \sqrt{1 + [f']^2}\, dx$$

$$f = \frac{1}{12}x^3 + x^{-1} \longrightarrow f' = \frac{1}{4}x^2 - x^{-2}$$

$$1 + (f')^2 = 1 + \left(\frac{1}{4}x^2 - x^{-2}\right)^2 = \frac{1}{16}x^4 + \frac{1}{2} + \frac{1}{x^4}$$

$$= 1 + \left(\frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{x^2}\right]^2$$

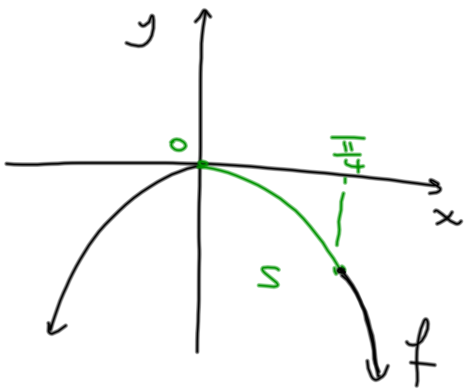
So $$S = \int_1^3 \sqrt{\left[\frac{1}{4}x^2 + \frac{1}{x^2}\right]^2}\, dx$$

$$= \int_1^3 \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right) dx = \left[\frac{1}{12}x^3 - \frac{1}{x}\right]_1^3$$

$$= \frac{9}{4} - \frac{1}{3} - \frac{1}{12} + 1$$

$$\boxed{= \frac{17}{6} \text{ unit}}$$

Eg: $f(x) = \ln(\cos x)$ over $\left[0, \frac{\pi}{4}\right]$



Find the length s

$f = \ln \cos x$, $f' = \frac{-\sin x}{\cos x} = -\tan x$

$1 + f'^2 = 1 + (-\tan x)^2$

$1 + f'^2(x) = \sec^2 x$

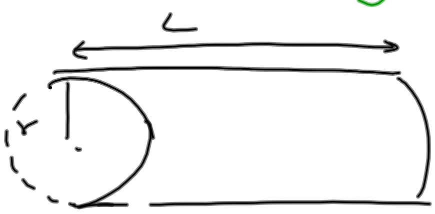
Now $s = \int_0^{\frac{\pi}{4}} \sqrt{1 + (f')^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} \sec x\, dx$

$= \Big[\ln \left| \sec x + \tan x \right| \Big]_0^{\frac{\pi}{4}}$

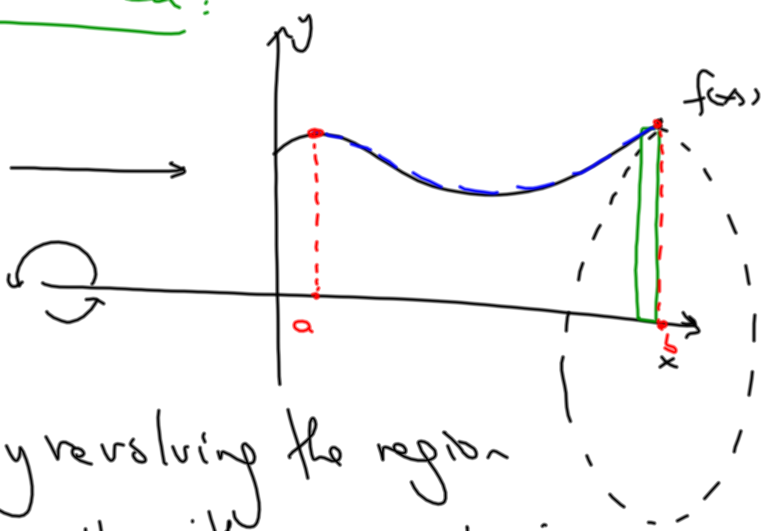
$= \ln\left(\left|\sqrt{2} + 1\right|\right) - \ln\left(\left|1 + 0\right|\right)$ $\quad \sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\boxed{s = \ln\left(\sqrt{2} + 1\right)} - 0 \simeq .8814$ unit

② Surface area:



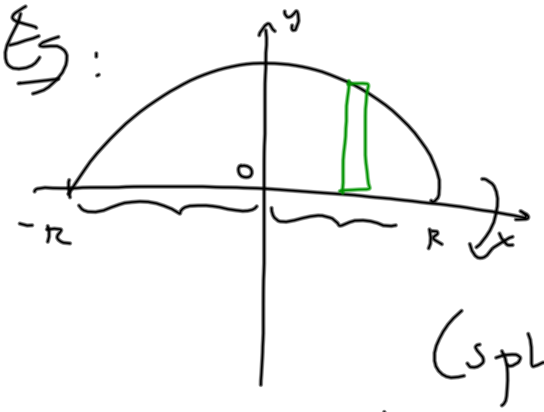
Surface area: $2\pi r L$



Surface Area obtained by revolving the region below f about the x-axis with $a \leq x \leq b$ is given by

$$\boxed{A = 2\pi \int_a^b \underbrace{f(x)}_{\text{radius}} \underbrace{\sqrt{1+\left(f'(x)\right)^2}}_{\text{length of } f} \, dx}$$

Eg:



Rotate $y=\sqrt{R^2-x^2}$

Find the surface area of the region (sphere) obtained by rotating $y=\sqrt{R^2-x^2}$ about x-axis!

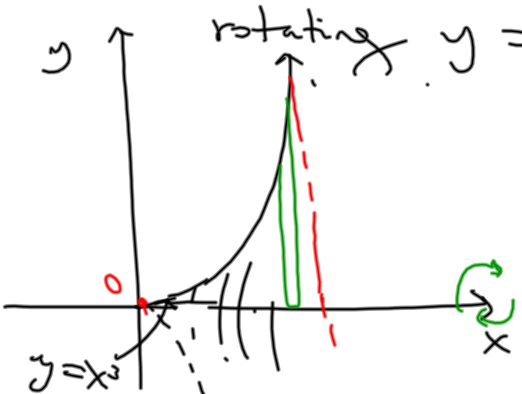
$$A = 2\pi \int_{-R}^{R} y\sqrt{1+(y')^2}\,dx = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}$$

$$= 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\,\frac{R}{\sqrt{R^2-x^2}}\,dx = 2\pi\int_{-R}^{R} R\,dx$$

$$= 2\pi\Big[Rx\Big]_{-R}^{R} = 2\pi\left(R^2+R^2\right) = \boxed{4\pi R^2 \text{ unit}^2}$$

Eg: $y = x^3$ with $x \in [0,1]$ over the x-axis.

Find the surface area obtained by rotating $y = x^3 \longrightarrow y' = 3x^2$



$$A = 2\pi \int_0^1 x^3 \sqrt{1+(3x^2)^2}\, dx$$

$$= 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\, dx$$

(u-substitution) Let $u = 1+9x^4 \longrightarrow du = 36x^3 dx$

$\longrightarrow \frac{1}{36} du = x^3 dx$

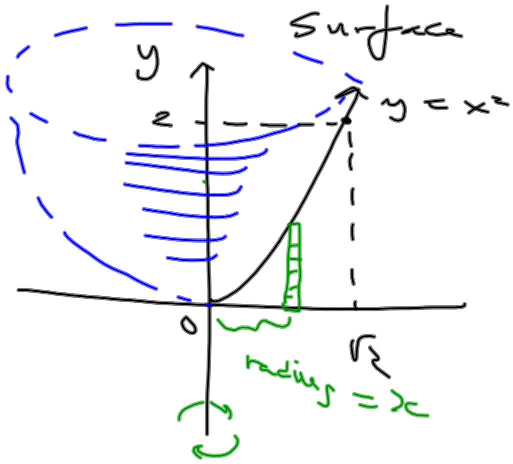
So $A = 2\pi \int \sqrt{u} \left(\frac{1}{36} du\right) = \frac{\pi}{18} \int u^{\frac{1}{2}} du$

$$= \frac{\pi}{18} \left(\frac{2}{3} u^{3/2}\right)$$

$$= \frac{\pi}{18} \left[\frac{2}{3} \sqrt{(1+9x^4)^3}\right]_0^1$$

$$= \frac{\pi}{27} \left[10^{3/2} - 1\right] \approx \boxed{3,563 \text{ unit}^2}$$

Eg: $y = x^2$, $x \in [0, \sqrt{2}]$, rotate the region y-axis and find the area of its surface



$$\text{Area} = 2\pi \int_0^{\sqrt{2}} \underbrace{x}_{\text{radius}} \sqrt{1 + [y']^2}\, dx$$

$$= 2\pi \int_0^{\sqrt{2}} x \sqrt{1 + [2x]^2}\, dx$$

$$= 2\pi \int_0^{\sqrt{2}} x \sqrt{1 + 4x^2}\, dx$$

$u = 1 + 4x^2$

$du = 8x\,dx \rightarrow \frac{1}{8} du = x\,dx$

$$A = 2\pi \int \sqrt{u} \left(\frac{1}{8} du\right) = \frac{\pi}{4}\left[\frac{2}{3} u^{3/2}\right]$$

$$= \frac{2\pi}{8} \int \sqrt{u}\, du = \frac{\pi}{6}\left[(1 + 4x^2)^{3/2}\right]_0^{\sqrt{2}}$$

$$\int k f\, dx = k \int f\, dx$$

$$= \frac{\pi}{6}\left[(1 + 8)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{\pi}{6}\left[\left(\sqrt[3]{9^2}\right) - 1\right] = \boxed{\frac{13\pi}{3} \text{ unit}^2}$$

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